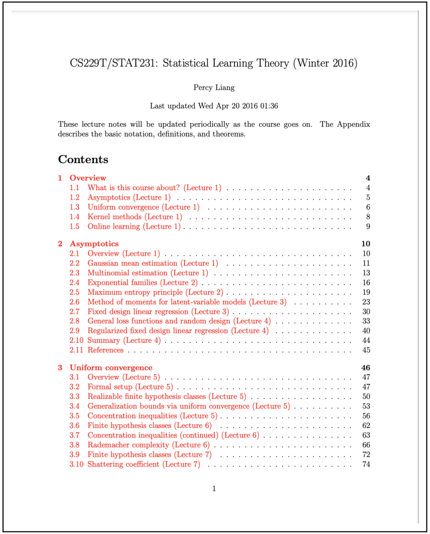
Trustworthy Machine Learning

Online Learning

Sangdon Park

POSTECH

Contents from



CS229T/STAT231: Statistical Learning Theory (Winter 2016)

Percy Liang

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These lecture notes will be updated periodically as the course goes on. The Appendix describes the basic notation, definitions, and theorems.

Contents

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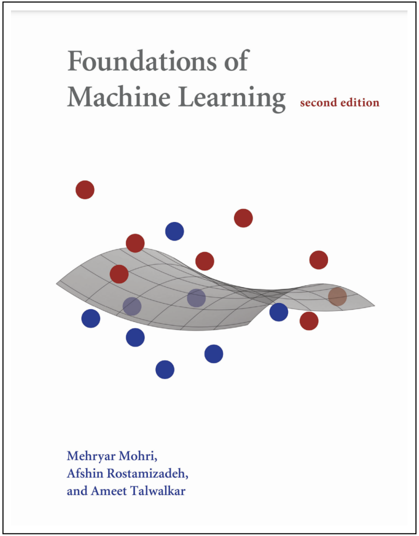

and various papers.

Motivation

- We have considered *statistical learning* (*i.e.*, learning under the i.i.d. assumption)

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- However, this assumption can be broken, *e.g.*, distribution shift, price data
- Here, we will weaken this assumption.
    - batch to online: “how data arrives”
    - statistical to adversarial: “how data are generated”

Setup

- Prediction task: learn to map an example $x \in \mathcal{X}$ to a label $y \in \mathcal{Y}$

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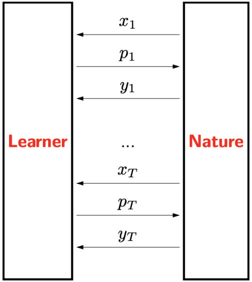
- Online learning game between a learner and nature



Protocol:

for $t = 1, \ldots, T$ **do**

Learner receives an example $x_t \in \mathcal{X}$

Learner outputs a prediction $p_t \in \mathcal{Y}$

Learner receives a true label $y_t \in \mathcal{Y}$

Learner suffers loss $\ell(y_t, p_t)$

Learner update model parameters

end for

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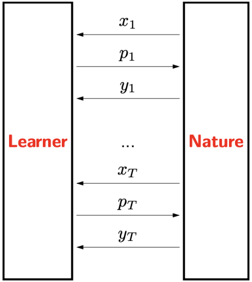

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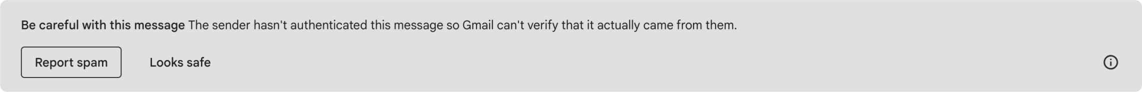
- The learner is a function $\mathcal{A}$ that returns the current prediction given the full history, *i.e.*,

$$p_{t+1} = \mathcal{A}(x_{1:t}, p_{1:t}, y_{1:t}, x_{t+1})$$

Example: Online Binary Classification for Spam Filtering



- examples: $\mathcal{X} := \{0,1\}^d$ are boolean feature vectors (presence or absence of a word)
- labels: $\mathcal{Y} := \{+1,-1\}$ are whether a document is spam or not
- zero-one loss: $\ell(y_t, p_t) = \mathbb{1}\,(y_t \neq p_t)$ is whether the prediction was incorrect

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  - This means that we can use any batch learning algorithm on the history at each time.
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- Online learning algorithms have the potential to adapt.
  - *e.g.*, we have labels on adversarial examples!
- For some applications (*e.g.*, spam filtering), examples are generated by an adversary.

"Goodness" Metric

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- No! In the adversarial setting, the adversary can manipulate data to make the learner trivially bad loss (*e.g.*, Nature can simulate Learner).
- What do you do when your grade is awful? Compare to the best grade in your class!

Regret

Definition

$$\text{Regret} := \underbrace{\sum_{t=1}^{T} \ell(y_t, p_t)}_{\text{learner}} - \underbrace{\min_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell(y_t, h(x_t))}_{\text{best expert}}$$

- $\mathcal{H}$ is a class of experts.
- The best export is a role model of the learner.
- We will consider the worst case regret (*i.e.*, labeled examples are generated by an adversary)

Negative Result

claim

For any deterministic learner $\mathcal{A}$, there exists an $\mathcal{H}$ and the sequence of labeled examples such that

$$\textit{Regret} \geq \frac{T}{2}.$$

- Too bad...
- Why? Prove under the following setup.
  - binary classification, *i.e.*, $y \in \{-1, +1\}$
  - zero-one loss, *i.e.*, $\ell(y_t, p_t) := \mathbb{1}(p_t \neq y_t)$
  - the learner is fully **deterministic**.

Negative Result: Why? Intuition

- An adversary (who has full knowledge of the learner) can choose y_t to make it different to the learner's choice p_t.
- Thus, the learner's cumulative loss is T!
- Not yet; how about the best expert's loss?
- Consider two experts, *i.e.*, $\mathcal{H} := \{h_{-1}, h_{+1}\}$ (where h_y always predict y).
- Thus, we have

$$\ell(y_t, h_{-1}(x_t)) + \ell(y_t, h_{+1}(x_t)) = 1 \quad \Rightarrow \quad \sum_{t=1}^{T} \ell(y_t, h_{-1}(x_t)) + \sum_{t=1}^{T} \ell(y_t, h_{+1}(x_t)) = T$$

$$\Rightarrow \sum_{t=1}^{T} \ell(y_t, h_{-1}(x_t)) \le \frac{T}{2} \quad \text{or} \quad \sum_{t=1}^{T} \ell(y_t, h_{+1}(x_t)) \le \frac{T}{2}$$

$$\Rightarrow \text{Regret} := \underbrace{\sum_{t=1}^{T} \ell(y_t, p_t)}_{=T} - \underbrace{\min_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell(y_t, h(x_t))}_{\le \frac{T}{2}} \ge \frac{T}{2}.$$

Outline

- Halving Algorithm
  - Deterministic
  - Separable assumption
  - Finite $\mathcal{H}$
- Exponential Weighting Algorithm
  - Randomized
  - No separable assumption
  - Finite $\mathcal{H}$
- Perceptron Algorithm
  - Deterministic
  - Separable assumption
  - Infinite $\mathcal{H}$

Add an Assumption without Randomization

Assumption (separable)

Assume that the best expert $h^* \in \mathcal{H}$ *obtains zero cumulative loss (i.e.,* $\ell(y_t, h^*(x_t)) = 0$ *for all* $t \in \{1, \dots, T\}$*).*

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- This impose restrictions on adversaries.
- We saw a similar assumption in PAC learning.
- Practical setup? adaptive conformal prediction

Halving Algorithm

Algorithm 1 Halving Algorithm

```
H_1 ← H
for t = 1, ..., T do
    Observe x_t
    Predict ŷ_t = MajorityVote(H_t, x_t)
    Observe y_t
    if ŷ_t ≠ y_t then
        H_{t+1} ← {h ∈ H_t | h(x_t) = y_t}
    else
        H_{t+1} ← H_t
    end if
end for
```

- $\mathcal{Y} := \{-1, +1\}$
- $\mathcal{H}_t$: a set of correct experts.
- Under the separable assumption, keep only correct experts.
- Due to the separable assumption, we can discard at least half of experts at some iterations!

Halving Algorithm: A Regret Bound

Theorem

Under the realizable assumption, for any $(x_t, y_t)_{t=1}^T$*, we have*

$$\textit{Regret} \leq \log_2 |\mathcal{H}|.$$

Halving Algorithm: A Regret Bound

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Under the realizable assumption, for any $(x_t, y_t)_{t=1}^T$*, we have*

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- Very strong results due to the separable assumption.
  - after a finite number of iterations, the predictor never makes mistakes.

Halving Algorithm: A Regret Bound

Proof Sketch

- Let M be the number of mistakes.
- For each mistake, at least half of the experts are eliminated, *i.e.*, if $\hat{y}_i$ made a mistake,

$$\frac{|\mathcal{H}_{i+1}|}{|\mathcal{H}_i|} \leq \frac{1}{2} \Rightarrow \frac{|\mathcal{H}_{T+1}|}{|\mathcal{H}|} \leq \frac{1}{2^M}.$$

- Due to the realizable assumption, we have

$$1 \leq |\mathcal{H}_{T+1}|.$$

- M = Regret.

Remove the Separable Assumption

- The separable assumption is too strong
- Let's remove this.
- Then, we need a randomization algorithm.
- One example: Exponential weighting algorithm.

Exponential Weighting Algorithm

Algorithm 2 Exponential Weighting Algorithm

$w_1 \leftarrow (1/|\mathcal{H}|, \ldots, 1/|\mathcal{H}|)$
for $t = 1, \ldots, T$ **do**
    Observe x_t
    Predict $\hat{y}_t = h^{i_t}(x_t)$, where $i_t \sim w_t$
    Observe y_t
    Update $w_{t+1}(i) \propto w_t(i) \exp\left\{-\eta \ell(h^i(x_t), y_t)\right\}$ for all $i \in \{1, \ldots, |\mathcal{H}|\}$
end for

- $\mathcal{H}$: a set of experts
- $\ell(\cdot) \in [0, 1]$

Exponential Weighting Algorithm

Algorithm 3 Exponential Weighting Algorithm

1: $w_1 \leftarrow (1/|\mathcal{H}|, \ldots, 1/|\mathcal{H}|)$
2: **for** $t = 1, \ldots, T$ **do**
3: Observe x_t
4: Predict $\hat{y}_t = h^{i_t}(x_t)$, where $i_t \sim w_t$
5: Observe y_t
6: Update $w_{t+1}(i) \propto w_t(i) \exp\left\{-\eta \ell(h^i(x_t), y_t)\right\}$ for all $i \in \{1, \ldots, |\mathcal{H}|\}$
7: **end for**

- $\mathcal{H}$: a set of experts
- $\ell(\cdot) \in [0, 1]$
- Due to the randomization in (4), an adversary cannot completely fool the learner.

Exponential Weighting Algorithm: A Regret Bound

Theorem

For any loss function ℓ with the range of $[0,1]$, we have

$$\textit{Regret} := \sum_{t=1}^{T} \mathbb{E}_{h\sim p_t} \ell(y_t, h(x_t)) - \min_{h\in\mathcal{H}} \sum_{t=1}^{T} \ell(y_t, h(x_t)) \leq \sqrt{T \ln |\mathcal{H}|}$$

if $\eta = \sqrt{\frac{8 \ln |\mathcal{H}|}{T}}$.

- No separable assumption.
- "learnable", *i.e.,* $\frac{\text{Regret}}{T} = \sqrt{\frac{\ln |\mathcal{H}|}{T}}$ with a mild assumption on loss.
- Still we assume a finite set of experts.

Exponential Weighting Algorithm I

Proof sketch

Definitions:

- $L_t^i := \sum_{s=1}^t \ell(h_i(x_s), y_s)$: the cumulative loss of h_i up to t
- $W_t := \sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta L_t^i\}$: a "potential value" at time t
- $W_0 := |\mathcal{H}|$: a "potential value" at time 0

Steps:

1. The lower bound of the "log-potential difference":

$$\ln \frac{W_T}{W_0} = \ln \sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta L_T^i\} - \ln |\mathcal{H}| \geq \ln \left(\max_{i \in \{1,\ldots,|\mathcal{H}|\}} \exp\{-\eta L_T^i\} \right) - \ln |\mathcal{H}| = -\eta \min_{i \in \{1,\ldots,|\mathcal{H}|\}} L_T^i - \ln |\mathcal{H}|.$$

Exponential Weighting Algorithm II

Proof sketch

2. The upper bound of the "log-potential difference":

$$\ln \frac{W_t}{W_{t-1}} = \ln \frac{\sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta L_t^i\}}{\sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta L_{t-1}^i\}} = \ln \frac{\sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta \ell(h_t^i(x_t), y_t)\} \exp\{-\eta L_{t-1}^i\}}{\sum_{i=1}^{|\mathcal{H}|} \exp\{-\eta L_{t-1}^i\}}$$

$$= \ln \mathbb{E}_{i_t \sim w_t} \exp\left\{-\eta \ell(h^{i_t}(x_t), y_t)\right\} \le -\eta \mathbb{E}_{i_t \sim w_t} \ell(h^{i_t}(x_t), y_t) + \frac{\eta^2}{8}$$

$$\Rightarrow \ln \frac{W_T}{W_0} \le -\eta \sum_{t=1}^{T} \mathbb{E}_{i_t \sim w_t} \ell(h^{i_t}(x_t), y_t) + \frac{\eta^2 T}{8}$$

- For any $s \in \mathbb{R}$ and a random variable $X \in [a, b]$, $\ln \mathbb{E} e^{sX} \le s\mathbb{E}X + \frac{s^2(b-a)^2}{8}$.

Exponential Weighting Algorithm III

Proof sketch

3. Combine the lower and upper bounds:

$$-\eta \min_{i\in\{1,\ldots,|\mathcal{H}|\}} L_T^i - \ln|\mathcal{H}| \leq -\eta \sum_{t=1}^{T} \mathbb{E}_{i_t\sim w_t}\ell(h^{i_t}(x_t), y_t) + \frac{\eta^2 T}{8} \Rightarrow$$

$$\sum_{t=1}^{T} \mathbb{E}_{i_t\sim w_t}\ell(h^{i_t}(x_t), y_t) - \min_{i\in\{1,\ldots,|\mathcal{H}|\}} L_T^i \leq \frac{\eta T}{8} + \frac{\ln|\mathcal{H}|}{\eta}$$

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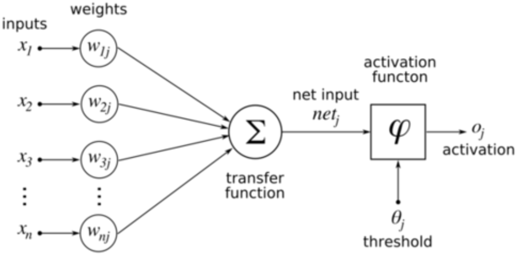
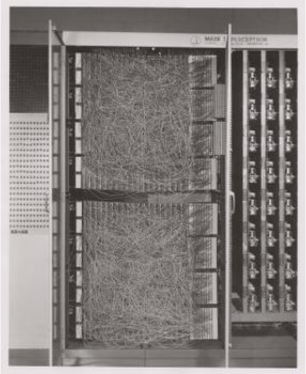
  - Deterministic
  - Separable assumption (with some margin)
  - Infinite $\mathcal{H}$

Perceptron: History

TLDR: Father of Neural Networks!



(a) Perceptron



(b) Mark I Perceptron machine

- Invented in 1943 by Warren McCulloch and Walter Pitts.
- Firstly implemented in 1958 by Frank Rosenblatt(!)

Perceptron Algorithm: Setup

- $\mathcal{D}$: change over time but require the separable assumption.
- $\mathcal{H}$: linear functions without bias terms – additional assumption
- ℓ: 0-1 loss for classification

Perceptron Algorithm

Algorithm 4 Perceptron Algorithm

$w_1 \leftarrow w_0 := 0$
for $t = 1, \ldots, T$ **do**
  Receives an example $x_t \in \mathcal{X}$
  $\hat{y}_t \leftarrow \mathrm{sign}(w_t \cdot x_t)$
  Receives a true label $y_t \in \mathcal{Y}$
  if $\hat{y}_t \neq y_t$ **then**
    $w_{t+1} \leftarrow w_t + y_t x_t$
  else
    $w_{t+1} \leftarrow w_t$
  end if
end for

Perceptron Algorithm: A Regret Bound

Theorem

Suppose $\|x_t\|_2 \leq r$ *for all* t *and for some* r, *and there exists* $\gamma > 0$ *and* $v \in \mathbb{R}^d$ *such that for all* t *we have*

$$\gamma \leq \frac{y_t(v \cdot x_t)}{\|v\|_2}.$$

Then, we have

$$\textit{Regret} \leq \frac{r^2}{\gamma^2}.$$

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- Assumption: a sequence is separable by a perfect classifier v with some margin

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Then, we have

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- Assumption: a sequence is separable by a perfect classifier v with some margin
- The bound does not depend on T

Perceptron Algorithm: A Proof Sketch

- Let $\mathcal{J} \subseteq \{1, \dots, T\}$ be the set of time indices when updated. Thus, Regret $= |\mathcal{J}|$.
- From the "margin" assumption, there exists γ and v for any $\mathcal{J}$ such that a margin of v from any mis-classified sample is larger than γ.

$$
\begin{aligned}
\overbrace{\gamma|\mathcal{J}|}^{\text{sum of margins}} &\le \frac{\sum_{t\in\mathcal{J}} y_t(v \cdot x_t)}{\|v\|} = \frac{v}{\|v\|} \cdot \sum_{t\in\mathcal{J}} y_t x_t \\
&\le \left\| \sum_{t\in\mathcal{J}} y_t x_t \right\| \\
&= \left\| \sum_{t\in\mathcal{J}} w_{t+1} - w_t \right\| = \|w_{T+1}\| = \sqrt{\|w_{T+1}\|^2} = \sqrt{\|w_{T+1}\|^2 - \|w_0\|^2} \\
&= \sqrt{\sum_{t\in\mathcal{J}} \|w_{t+1}\|^2 - \|w_t\|^2} = \sqrt{\sum_{t\in\mathcal{J}} \|w_t + y_t x_t\|^2 - \|w_t\|^2} = \sqrt{\sum_{t\in\mathcal{J}} 2y_t w_t \cdot x_t + \|x_t\|^2} \\
&\le \sqrt{\sum_{t\in\mathcal{J}} \|x_t\|^2} \le \sqrt{\sum_{t\in\mathcal{J}} r^2} = r\sqrt{|\mathcal{J}|}.
\end{aligned}
\tag{1}
$$

- (1): Cauchy-Schwarz inequality, *i.e.*, $u \cdot v \le \|u\|\|v\|$

Conclusion

- What we learned
  - Halving Algorithm
    - Deterministic
    - Separable assumption
    - Finite $\mathcal{H}$
  - Exponential Weighting Algorithm
    - Randomized
    - No separable assumption
    - Finite $\mathcal{H}$
  - Perceptron Algorithm
    - Deterministic
    - Separable assumption
    - Infinite $\mathcal{H}$
- Interesting materials
  - Online convex optimization
  - Stochastic bandits
  - Adversarial bandits